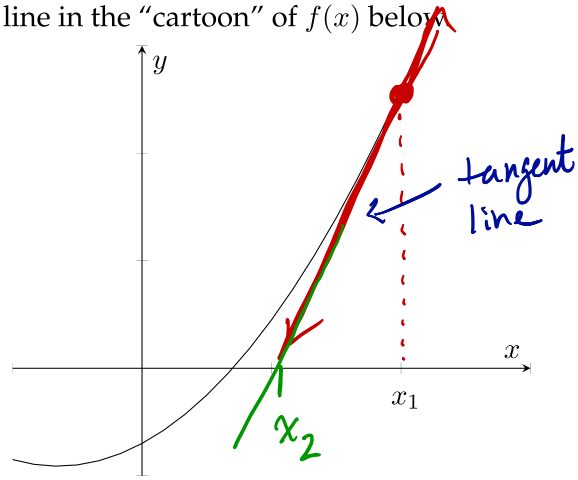
Lecture Notes: 4-8 Newton's Method (part 1)

MOTIVATING QUESTION: Recall that we wanted to find the x-intercepts of $f(x) = x - 2\sin x$. From the graph we knew there exists a positive (and negative) solution. How to find it?

WARM-UP PROBLEMS:

1. Write the equation of the line tangent to the curve $y = f(x)$ at the x-value x_1. Sketch the tangent line in the "cartoon" of $f(x)$ below.



$m = f'(x_1)$ slope

$(x_1, f(x_1))$ point

line: $y - f(x_1) = f'(x_1)(x - x_1)$

or $\boxed{y = f'(x_1)(x - x_1) + f(x_1)}$

2. In your picture above, label the x-value where the tangent line intersects the x-axis as x_2.

3. Solve for x_2 using your equation from part (1) above.

x_2 occurs when $y=0$. Set $y=0$ & solve for x. here

$$0 = f'(x_1)(\boxed{x} - x_1) + f(x_1)$$

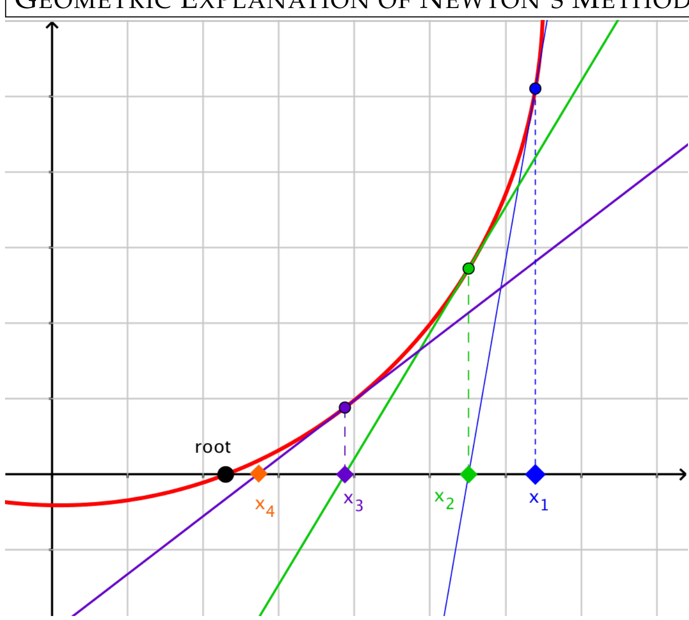
$$-f(x_1) = f'(x_1)\boxed{x} - x_1 f'(x_1)$$

$$x_1 f'(x_1) - f(x_1) = f'(x_1) \cdot \boxed{x}$$

$$\boxed{x_2} = \frac{x_1 f'(x_1) - f(x_1)}{f'(x_1)} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

observe, this is actually a really simple expression.

GEOMETRIC EXPLANATION OF NEWTON'S METHOD:



Algebraically

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

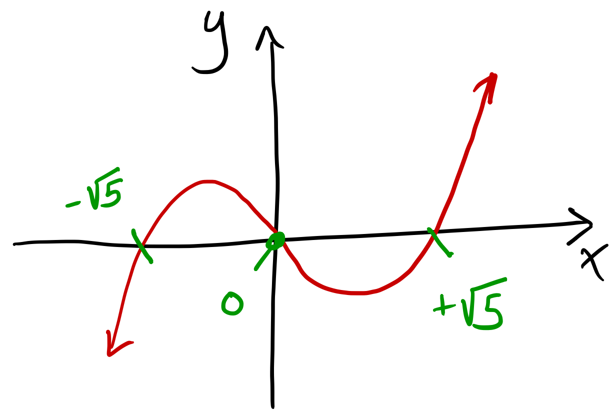
* There is nothing special about $x_1 \to x_2$.

MODEL PROBLEM: Let $f(x) = x^3 - 5x$

1. Factor $f(x)$, find its roots algebraically, and sketch its graph.

$f(x) = x(x^2-5) = x(x+\sqrt{5})(x-\sqrt{5})$

roots $x = 0, -\sqrt{5}, +\sqrt{5}$



2. Assume you couldn't factor the function and you wanted to find its positive root. What would be a reasonable first guess and why?

$x=3,\ f(3) = 27-15 = 12 > 0$

$x=2,\ f(2) = 8-10 = -2 < 0$

So there's a positive root between $x=2$ and $x=3$.

I'd pick $\boxed{x=2.5}$.

3. Using a first guess of $x_1 = 3$, calculate 2 iterations of Newton's method

Recall $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $f(x) = x^3-5x$, $f'(x) = 3x^2-5$

If $x_1 = 3$, then $x_2 = 3 - \dfrac{f(3)}{f'(3)} = 3 - \dfrac{27-15}{27-5} = 3 - \dfrac{12}{22} \approx 2.45454545$ 1 iteration

If $x_2 = 2.45454545$, then $x_3 = (2.45454545) - \dfrac{f(2.45454545)}{f'(2.45454545)} \approx 2.26215377$

4. How close is your estimate of the root, x_3, to the actual root?

$\sqrt{5} \approx 2.2360679 8$. Difference: less than 0.0260 not too shabby.